Guiding Children's Learning of

Mathematics

13e

ART JOHNSON / STEVE TIPPS / LEONARD M. KENNEDY

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For Clover Elise Ingersoll,
who is helping her grandfather
rediscover the enchantment
that is mathematics.
—A. J.

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PREFACE

The ecosystem of mathematics education is in the midst of systemic change. There are two factors leading this change: curriculum developments and technology. Both revolutions have been going on for several decades and show no sign of slowing or ending. Anyone who has been teaching mathematics for the past few decades has had a front row seat for these momentous changes.

When I first began teaching, the mathematics curriculum was essentially whatever textbook you were using. I was using Mary Dolciani's book, the one that sparked the New Math wars, and *Stein's Refresher*, which was essentially a drill-and-kill textbook. If I stayed the course with these texts, I was usually spared any administrative intervention or parental questions. Occasionally, a problem would arise whenever a student transferred into my class from another state or even another town. The student might have taken math in grade 6 and even used the same text, but the student's math experiences bore no resemblance to what my grade 6 class was studying. It was very hard to know if the new student was far advanced or far behind my class.

In 1989, the National Council of Teachers of Mathematics released *Curriculum and Evaluation Standards for School Mathematics,* an outline of mathematics concepts and standards for the entire public school grade. It was welcomed by most mathematics teachers but critiqued by others because of a lack of focus on number facts. In 2000, NCTM published the *Principles and Standards for School Mathematics*, which attempted to answer the critics' concerns. However, few school districts adopted the Standards, and no states did. Common Core State Standards was to change all that.

The other aspect of the revolution is technology. When I first began teaching, the technological developments were the handheld calculator and the overhead projector. My school bought two projectors, and the entire school had to share them. I went all in for the handheld calculator in my class. I spent $89.00 back then for a *Bowmar Brain*, a calculator that performed the four arithmetic operations, plus it had a % key. In addition, if I plugged it in overnight, I had 1 hour of power the next day. And it came with its own genuine imitation leather case.

Next to appear were personal computers, and soon most schools had either a room full of computers or a classroom computer for each teacher. Still, the technology we enjoy today was slow to advance. As recently as 2002, I accompanied my wife to a conference in Vancouver. While she attended conference sessions, I worked on a writing project and kept in touch with my colleagues at Boston University via e-mail. It wasn't as simple as it sounds. There were no smartphones or WiFi. I had to go to the local library and sign up for an hour of computer time. I had to use their computers to send and receive e-mails.

I write the above not to demonstrate how difficult teaching was some years ago, but to suggest that the changes we see today have forever altered the face of mathematics teaching.

Despite all the political rhetoric surrounding Common Core, it is here to stay. Although some states have reconsidered their adoption and others have never accepted the Common Core curriculum, a large majority of states are staying the course with Common Core. Common Core is not an official national curriculum. Yet every mathematics publisher has designed their textbooks, ancillary materials, and support materials around the approaches outlined in Common Core. Thus, for all intents and purposes, Common Core is a *de facto* national curriculum. Even a state that has not adopted Common Core will nevertheless use mathematics materials from major publishers that will generally align with Common Core. Now students who change school districts in the middle of the year will be able to step into their new mathematics class with little difficulty. In sum, we now have the national curriculum many mathematics educators have long wanted.

Another feature of Common Core is the focus on mathematics processes. Mathematics educators and authors have generally focused on content rather than process. Mathematical processes are more demanding to discern, harder to model, and difficult to include in effective problems and activities. The Mathematical Practices of Common Core illuminate and imbed into the curriculum eight critical processes for learning and doing mathematics. The eight Mathematical Practices specify and describe those actions and attitudes that produce mathematics thinking and problem solving.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Chapter 1 has more information about the Common Core Mathematics Standards and the Mathematical Practices.

The technology revolution has been even more astounding. Computers are no longer a factor in the classroom. Tablets and smartphones are doing all the heavy lifting. Students no longer need a personal calculator; they can simply download a free calculator app onto their personal device. In the classroom, the Smartboard has more than replaced the blackboard of the last century. The Smartboard has become an interactive tablet for students in ways we have not yet fully integrated. Students now use podcasts, social media, and original videos as a regular part of their mathematics learning.

And the future? It is difficult to predict. The days of textbooks may be numbered, as more and more offerings are all electronic. No more lost, worn-out, or dated textbooks. No need to lug a full backpack between home and school. Anything in today's textbooks is available online to a tablet or smart phone. A new invention that is just making its way into schools is the 3D printer. How will 3D printers impact mathematic education? We will know as educators explore and work with these new devices. You can be sure that their impact will again change mathematics education, as did the calculator, tablets, and smartphone.

—Art Johnson

Organization and Guiding Principles

Any text that claims to equip preservice teachers to teach mathematics in elementary school must address the curriculum advances in mathematics education and the effects of new and emerging technologies. *Guiding Children's Learning of Mathematics* is designed to do just that. As described in Chapter 1, this text is based on eight Mathematics Teaching Practices found in *Principles to Actions: Ensuring Mathematical Success for All*, published in 2014 by the National Council of Teachers of Mathematics.

Guiding Children's Learning of Mathematics, 13th edition, also draws on core beliefs about how teachers help children develop as mathematical thinkers and consumers:

1. Teachers engage children in active learning—physically, emotionally, and cognitively.
2. Teachers encourage students to reflect on their experiences and to construct meaning.
3. Teachers invite children to think at higher cognitive levels.
4. Teachers help children connect mathematics to their lives.
5. Teachers encourage children to communicate their ideas in many forms and settings.
6. Teachers constantly monitor and assess students' understanding and skill.
7. Teachers adjust their instruction to fit the needs, levels, and interests of children.
8. Teachers create a positive learning environment that supports critical and creative thinking.

Using the Common Core and NCTM *Principles and Standards* as a starting place, the 13th edition provides mathematical foundations and content background illustrated through lessons, activities, and suggestions about how to create an active classroom for learning and thinking.

Text Organization

This 13th edition is organized in two parts. In the first part, the Common Core is introduced as a foundation for how teachers think and make decisions about their practice. The Common Core presents a comprehensive mathematics curriculum through subject *domains* such as Geometry or Measurement. Within each domain are specific standards for each grade level. These standards detail the topics appropriate for a domain at that grade level.

The first part also considers critical principles for how teachers think and make decisions about their practice:

- *Curriculum:* Do all children receive a well-rounded and balanced program in mathematics?
- *Equity:* Do all children have access and opportunities to be successful in mathematics?
- *Learning:* Are the methods I use based on what is known about how children learn?
- *Teaching:* Do the methods I use enhance learning by engaging children in mathematical thinking, developing concepts and skills, and applying their knowledge to engaging problems?
- *Assessment:* Do I use assessment to determine children's strengths and needs on a continuous basis and adjust my instruction accordingly?
- *Technology:* Do I use technology to help children explore and learn mathematical concepts?

Each principle is defined and illustrated with examples for implementation. When teachers understand the context for their instructional decisions, they can make better ones. New and revised content in Part 1 include:

- Description and use of Common Core State Standards and Mathematical Practices
- Additional suggestions for working with culturally and linguistically diverse children
- New technologies such as apps and interactive Internet sites and their use for instruction
- More emphasis on diagnostic assessment in a revised chapter on assessment

Part 2 in the textbook emphasizes the mathematical curriculum and strategies needed for teaching integrated Common Core Standards in the following domains:

- Numbers and Operations
- Geometry
- Operations and Algebraic Thinking
- Measurement and Data
- Ratio and Proportional Relationships
- Statistics and Probability

After an introductory chapter on problem-solving strategies, each content standard is addressed in one or more chapters. Chapters that present foundational concepts and skills in primary grades K–2 are labeled *developing*. Chapters emphasizing content and skills taught in grades 3–5 and building on earlier ideas are labeled *extending*. This structure shows the alignment of mathematics over the K–5 curriculum.

Text Features

Within each content chapter, a variety of activities, lessons, and vignettes demonstrate how teachers engage students in learning mathematics. Features in Part 2 include:

- *Activities:* This feature highlights lessons and activities that introduce and reinforce the standards and are easy for preservice and in-service teachers to implement in the classroom. Examples include lessons, games, and activities using manipulatives, technology such as apps, take-home lessons, and children's literature. The earlier List of Activities correlates each activity to its location and the associated standards so relevant activities can be easily located. All activities are available as Professional Resource Downloads with MindTap. (See the Supplements listing on page xxiv for more information about MindTap.)
- *Correlation to the Common Core State Standards (CCSS):* Correlations to the CCSS are integrated throughout. Relevant CCSS to the chapter are listed in abbreviated form at the beginning of each chapter, and each activity is correlated to the CCSS. A Standards Correlation Grid on the inside front and back covers allows you to quickly locate the relevant standards in each chapter.

- *Learning Objectives:* Learning objectives are provided at the beginning of each chapter and now correlate with main headings within the chapter and the summary at the end of the chapter. The objectives highlight what students need to know to process and understand the information in the chapter. After completing the chapter, students should be able to demonstrate how they can use and apply their new knowledge and skills.
- *Exercises:* Short exercises appear at the end of some sections to provide a pause to reflect on a key concept of the section content.
- *CLD Alert:* This feature provides tips to help inservice teachers support culturally and linguistically diverse (CLD) students, both through identifying specific challenges that CLD students may experience and ways that teachers can integrate and appreciate the cultural heritage of all students in the curriculum.
- *Misconception:* This feature highlights common misunderstandings students have about specific math operations and concepts, which may also help preservice teachers clarify and correct their own misconceptions.
- *Children's Bookshelf:* A list of children's literature titles relevant for exploring the chapter with students are included at the end of the chapters in Part 2.
- *Black-Line Masters:* Additional resources to support activities are provided in Appendix C, also available as Professional Resource Downloads in MindTap. Examples include hundreds charts, fraction circles, tangram templates, and more.

Accompanying Teaching and Learning Resources

This 13th edition of *Guiding Children's Learning of Mathematics* is accompanied by an extensive package of instructor and student resources.

MindTap™: The Personal Learning Experience

MindTap for Johnson/Tipps/Kennedy, *Guiding Children's Learning of Mathematics*, 13th edition, represents a new approach to teaching and learning. MindTap is a highly personalized, fully customizable learning platform with an integrated eportfolio. It helps students to elevate thinking by guiding them to:

- Know, remember, and understand concepts critical to becoming a great teacher;
- Apply concepts, create curriculum and tools, and demonstrate performance and competency in key areas in the course, including national and state education standards;
- Prepare artifacts for the portfolio and eventual state licensure, to launch a successful teaching career; and
- Develop the habits to become a reflective practitioner.

As students move through each chapter's Learning Path, they engage in a scaffolded learning experience, designed to move them up Bloom's Taxonomy, from lower- to higher-order thinking skills. The Learning Path enables preservice students to develop these skills and gain confidence by:

- Engaging them with chapter topics and activating their prior knowledge by watching and answering questions about authentic videos of teachers teaching and children learning in real classrooms;
- Checking their comprehension and understanding through "Did You Get It?" assessments, with varied question types that are autograded for instant feedback;
- Applying concepts through mini-case scenarios—students analyze typical teaching and learning situations, and then create a reasoned response to the issue(s) presented in the scenario; and
- Reflecting about and justifying the choices they made within the teaching scenario problem.

MindTap helps instructors facilitate better outcomes by evaluating how future teachers plan and teach lessons in ways that make content clear and help diverse students learn, assessing the effectiveness of their teaching practice, and adjusting teaching as needed. MindTap enables instructors to facilitate better outcomes by:

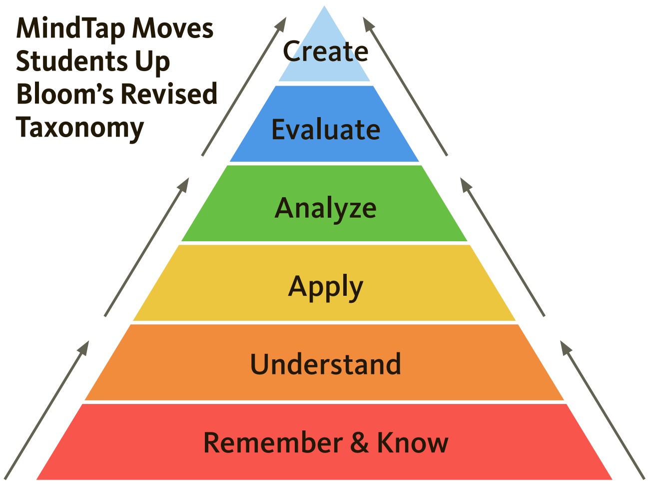
- Making grades visible in real time through the Student Progress App so students and instructors always have access to current standings in the class.

- Using the Outcome Library to embed national education standards and align them to student learning activities, and also allowing instructors to add their state's standards or any other desired outcome.
- Allowing instructors to generate reports on students' performance with the click of a mouse against any standards or outcomes that are in their MindTap course.
- Giving instructors the ability to assess students on state standards or other local outcomes by editing existing or creating their own MindTap activities, and then by aligning those activities to any state or other outcomes that the instructor has added to the MindTap Outcome Library.

MindTap for Johnson/Tipps/Kennedy, *Guiding Children's Learning of Mathematics*, 13th edition, helps instructors easily set their course since it integrates into the existing Learning Management System and saves instructors time by allowing them to fully customize any aspect of the Learning Path. Instructors can change the order of the student learning activities, hide activities they don't want for the course, and—most importantly—create custom assessments and add any standards, outcomes, or content they do want (e.g., YouTube videos, Google docs). Learn more at **www.cengage.com/mindtap**.

Instructor's Manual

An online Instructor's Manual accompanies this book. It contains information to assist you in designing the course, including sample syllabi, discussion questions, teaching and learning activities, field experiences, learning objectives, and additional online resources.



Test Bank

For assessment support, the Test Bank includes multiple-choice and short-answer questions for each chapter.

PowerPoint Lecture Slides

These vibrant Microsoft PowerPoint lecture slides for each chapter assist you with your lecture by providing concept coverage using images, figures, and tables directly from the textbook.

Cognero

Cengage Learning Testing Powered by Cognero is a flexible online system that allows you to author, edit, and manage test-bank content from multiple Cengage Learning solutions; create multiple test versions in an instant; and deliver tests from your LMS, your classroom, or wherever you want.

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1

Elementary Mathematics for the 21st Century

Elementary mathematics has been the subject of much discussion, debate, and controversy in recent years. At the center of this debate is whether children should focus on basic computation skills or develop a wider range of knowledge and skills in mathematics. The curriculum recommended by the Common Core State Standards for Mathematics (2009) and adopted by many states emphasizes thinking and problem solving related to all mathematical topics: numbers and operations, statistics, measurement, probability, geometry, and algebra. The content is connected to living, working, and solving problems in a technological and information-based society. Computational skills are still somewhat important, but students must know when and how to use numbers to solve problems.

Just as the focus of mathematics has changed toward a balanced curriculum, recommended teaching strategies reflect new understandings of how students learn based on cognitive research. Teachers and parents are challenged to consider mathematics differently from their school mathematics experience, which was dominated by calculations and procedures, drill and repetition, memorization and rote applications, and solitary work. An ideal classroom today finds students working together on challenging problems related to their lives, explaining their thinking to each other and their teacher, and using a variety of materials and technologies to show what they understand and can do.

LEARNING OBJECTIVES

After reading this chapter, you will be able to:

- **1-1** Identify problem solving as the central idea in school mathematics.
- **1-2** Understand how the Common Core State Standards, Mathematics Teaching Practices, and Essential Elements are foundations of a successful mathematics school program.
- **1-3** Define how Common Core and *Principles to Actions* are being implemented in the classroom.

1-1 Solving Problems Is Basic

Many adults believe that they are not competent in mathematics; some even have anxiety just thinking about it. At the same time, these people use mathematics in their daily lives without much difficulty when they shop, cook, manage their money, work on home improvement projects, or plan for travel. One goal of a balanced curriculum is to provide all citizens with mathematical concepts and skills they will need in life: for budgeting and saving, financing a house or car, calculating a tip at a restaurant, or estimating distances and gas mileage.

> EXERCISE 1.1

Is a regular box of cereal for $3.75 a better buy than the smaller box for $2.75 or the giant box for $4.75 (Figure 1.1)? Does having a "50 cents off" coupon change the answer? Using the information in Figure 1.1, work in groups of two or three to decide which cereal to buy. What else would you consider when deciding which cereal to buy?

A numerical answer is only one factor to consider in a daily decision. Problem solving involves other issues that may be more important than the computed answer. Reasoning is needed to decide how much cost, taste, and nutrition are considered in the final decision. Even after complex calculations, such as the cost of remodeling or the various incentives offered for buying a car, the numerical answer is only one of many other factors involved.

Adults, even those who believe they were not good in school mathematics, often develop mathematical skills in their jobs. Carpenters and contractors measure accurately and estimate job costs

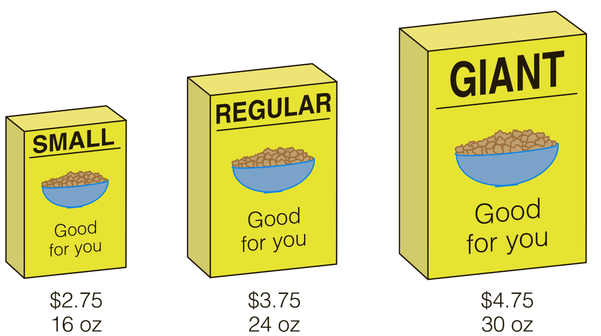

Figure 1.1 Which cereal would you buy?

and materials. Accountants, graphic designers, and hospital workers use calculators and computers routinely to record and analyze information and designs. When mathematics is applied to real-world situations, many adults find that they *are* capable in mathematics, despite their negative attitudes toward mathematics in school. The need to connect mathematics to real-world situations has been one motivation for reform of mathematics curriculum.

The vision for teaching mathematics has also changed. The old idea was that mathematics was a solitary activity done primarily with worksheets. Now teachers ask students to work together to solve interesting problems, puzzles, games, and investigations, using new and emerging technologies. When solving these problems, students develop the concepts, skills, and attitudes needed for life and work. Elementary mathematics teachers design classroom events that provide meaningful mathematical learning experiences.

- If juice boxes are packaged in groups of three, how many packs are needed for 20 children?
- How much money is needed to buy lunch, a snack, and a pencil at the school store?
- Is January a good month to take a field trip to the zoo? Why or why not?
- Will this 12 × 15 inch piece of paper be large enough to cover a cube for my art project?
- How many children prefer hamburgers to pizza? hamburgers to hot dogs? pizza to hot dogs?

Students determine what information they need and how to use it to solve problems. Problem-focused teachers probe further: "Is there only one answer? Can anyone see another way to solve this problem?"

As students engage problems with multiple answers and multiple solution paths, they become more flexible in their thinking. When problems serve as the context of teaching, children ask, "Does this answer make sense?" Students learn as they write, draw, act, read about, and model their thinking. They engage in dialogue, demonstration, and debate about mathematical ideas. When children are actively engaged in doing mathematical tasks, they are thinking about how mathematics works. This new vision of mathematics includes a balanced mathematics program for students of all ages that focuses on concepts, processes, and applications, with problem solving at its core.

1-2 A Comprehensive Vision of Mathematics

In 1989 the National Council of Teachers of Mathematics (NCTM) published its seminal work *Curriculum and Evaluation Standards for School Mathematics*. This document marked the first time any national educational organization published content standards in any subject for school districts to use when developing their local educational programs. The Standards provoked much discussion because they were designed for adoption across the United States, something that had never happened before. The Standards lacked any official standing before any state boards of education or any local school districts. Of further concern by many was that any attempt to adopt the same mathematics program and standards nationwide might in some way usurp local control of students' education. As a result, few school districts adopted the Standards as a guide to their mathematics program. Additionally there were a number of criticisms about the Standards' lack of rigor regarding basic number facts and algorithms.

1-2a Revised NCTM Standards

Responses to the criticisms of the Standards were incorporated by NCTM in a more recent document, *Principles and Standards for School Mathematics* (2000). These revised Standards recognized the need for students to master basic mathematics content and procedures, including the importance of computation. These Standards also recognized the importance of new and emerging technologies on the mathematics classroom. This document enjoyed wider acceptance than the previous Standards. Still, these revised Standards did not have the standing to be adopted by districts. Publishers, however, were careful to incorporate the Standards in their textbooks, so that the new Standards became widely adopted into local school districts via their textbook adoptions. In brief, the 2000 Standards became a de facto national mathematics curriculum. To be sure, some mathematics texts still focused on "basic" math, with a concentration on computation. Yet, even these traditional programs followed the outline of the Standards, if not the spirit.

The NCTM Standards also included five **process standards**, which identified mathematical process for the first time in a national document. The process standards are:

- Problem solving
- Reasoning and proof
- Communications
- Connections
- Representation

In 2006 NCTM published *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics*. This document differed from the Standards in a very specific way. The Standards listed concepts and skills within grade bands: K–2, 3–5, 6–8. The Curriculum Focal Points are an example of how to answer the question, "What are the key mathematical ideas or topics on which a mathematics curriculum can build?" According to the NCTM (2006), "the Curriculum Focal Points are intended to address curriculum, or *what* is taught, rather than instruction or *how* it is taught." So, the Curriculum Focal Points show stronger connections between concepts and a given grade level, and clarify how a topic such as division would develop across different grade levels.

1-2b Common Core State Standards for Mathematics

By 2009 several trends in the mathematics education of the nation came to the attention of the National Governors Association (NGA). U.S. students' mathematics scores on international assessments were consistently in the bottom half of test scores. The mathematics curriculum across the country was a confusing hodgepodge of topics and content that had no rational organizing structure. Students in one community might study an algebra I course that was entirely different from an algebra I course in a neighboring town or across a state line. Finally, students who entered postsecondary education frequently required remedial mathematics courses to do college level work, despite getting good grades in high school. NGA initiated a task force to look further into these trends and to address the criticism of the nation's mathematics curriculum as being "a mile wide and an inch deep" (Governors Education Symposium, 2009).

Following the task force report, the NGA—along with the Council of Chief State School Officers

<table>
<tr><th>Grade K</th><th>Grade 1</th><th>Grade 2</th><th>Grade 3</th><th>Grade 4</th><th>Grade 5</th><th>Grade 6</th><th>Grade 7</th><th>Grade 8</th></tr>
<tr><td colspan="9">Geometry</td></tr>
<tr><td colspan="6">Measurement and Data</td><td colspan="3">Statistics and Probability</td></tr>
<tr><td colspan="6">Number and Operations in Base 10</td><td colspan="3">The Number System</td></tr>
<tr><td colspan="6">Operations and Algebraic Thinking</td><td colspan="3">Expressions and Equations</td></tr>
<tr><td>Counting and Cardinality</td><td colspan="2"></td><td colspan="3">Number and Operations—Fractions</td><td colspan="2">Ratios and Proportional Relationships</td><td>Functions</td></tr>
</table>

Figure 1.2 Learning trajectories in CCSSM

(CCSSO) and the nonprofit education reform group Achieve—came together to make sure the goals of the report became a reality. It was decided that "the key to advancing any of these recommendations was to start with the standards" (Bidwell, 2014).

While the effort was spearheaded by the NGA, the CCSSO, and Achieve, representatives from other national organizations were also enlisted for their input, such as the National Council of Teachers of Mathematics and members of both the American Federation of Teachers and the National Education Association—the two largest teachers' unions in the country. The NGA and the CCSSO also received nearly 10,000 comments on the new Standards during two public comment periods.

The result was the Common Core State Standards for Mathematics (CCSSM, 2009). This document specified what topics and concepts were appropriate at each grade level, and provided easy to follow learning trajectories for various mathematics topics, such as the development of fractions (see Figure 1.2). In addition to detailed standards (of which there are 21–28 for each grade from kindergarten to eighth grade), the Standards present an overview of "critical areas" for each grade. (See the entire document in **Appendix A**.)

Unlike the Standards documents published by NCTM, CCSSM has the backing of state organizations, and the curriculum was adopted by 45 states and the District of Columbia as the official curriculum of each state. There has been some continued resistance to the Common Core for various reasons, none related to the mathematics of the document. Some fear federal intrusion in local education decisions and so oppose a national program like Common Core. Others object to the cost of instituting a new mathematics curriculum. Finally, there is a strong concern about the testing aspect of assessment that has been tied to Common Core.

A new feature of Common Core is the Standards of Mathematical Practices. These echo the Content Standards of NCTM and add more detail and application. The Standards of Mathematical Practices clearly and specifically detail how each practice can be included in a mathematics class. The Standards of Mathematical Practices are given in Table 1.1.

TABLE 1.1 Standards of Mathematical Practices of Common Core

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

SOURCE: CCSSM (2010).

Chapter 2 and the content chapters in the second half of the text will detail specific parts of the Common Core domains and explain how the Standards of Mathematics can be included into typical classroom lessons and activities.

NCTM, an international professional organization of more than 100,000 professionals in PK–16 mathematics education, provides resources and guidance to teachers, schools, school districts, and state and national policymakers through policy recommendations, conferences, and publications.

1-2c Principles of Mathematics Teaching

NCTM continues to lead mathematics education reform and development. In 2014, NCTM published *Principles to Actions: Ensuring Mathematical Success for All*. The focus here is teaching and teachers. The *Principles to Actions* describes eight Mathematics Teaching Practices that effective mathematic teachers could use to frame their teaching with new and emerging trends in mathematics education (see Table 1.2 and Figure 1.3). These Mathematics Teaching Practices concentrate on the *how* of teaching the new mathematics programs, in contrast to Common

TABLE 1.2 Mathematics Teaching Practices

1. ESTABLISH MATHEMATICS GOALS TO FOCUS LEARNING.

Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.

2. IMPLEMENT TASKS THAT PROMOTE REASONING AND PROBLEM SOLVING.

Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.

3. USE AND CONNECT MATHEMATICAL REPRESENTATIONS.

Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.

4. FACILITATE MEANINGFUL MATHEMATICAL DISCOURSE.

Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

5. POSE PURPOSEFUL QUESTIONS.

Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.

6. BUILD PROCEDURAL FLUENCY FROM CONCEPTUAL UNDERSTANDING.

Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.

7. SUPPORT PRODUCTIVE STRUGGLE IN LEARNING MATHEMATICS.

Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.

8. ELICIT AND USE EVIDENCE OF STUDENT THINKING.

Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

SOURCE: CCSSM (2010).

Figure 1.3 Eight Mathematics Teaching Practices

Core focus on the specifics of the mathematics in the programs, or the *what*.

The details of these Mathematics Teaching Practices begin with commonly accepted facts about mathematics education, but then advance beyond these basic principles to develop a mathematics teaching practice that will effectively deal with the many factors that are an integral part of mathematics education from equity and minority issues to fluency and mathematics dialogue.

1. ESTABLISH MATHEMATICS GOALS TO FOCUS LEARNING Clear goals for teaching seem self-evident, but in the case of mathematics it is critical that, as concepts are developed, the goals reach across grades, so there is a seamless flow from one level of understanding to another. A lesson on multiplying fractions cannot exist independently from the preceding and following lessons dealing with fractions. The lesson has to be situated in a comprehensive and coherent development of fractions that is in place across many grade levels. The specifics of the fraction lesson and its position in the development of fractions concepts depend on the overarching goals determined for fraction learning.

Clear explicit goals benefit teachers as they work to develop lessons and activities that exist within mathematics learning progressions. Every lesson should be written to support the goals for the unit at hand and for more far-reaching goals for that grade, and ultimately for the entire concept across several grades. Students who learn in a classroom with clearly established goals that are shared with them perform better than students who are not informed about those goals (Daro, Mosher, & Corcoran, 2011).

2. IMPLEMENT TASKS THAT PROMOTE REASONING AND PROBLEM SOLVING Although students have always solved problems in mathematics, frequently there were problems that required students to recall a specific formula or algorithm. Problems that simply demand the application of an algorithm or formula are low-level problems. No reasoning about mathematics is required to solve them. High-level problems require students to reason about the problem itself and then to think about an appropriate strategy. High-level problems "actively engage students in reasoning, sense making, and problem solving so that they develop a deep understanding of mathematics" (NCTM, 2014).

Compare the low-level and high-level problems below.

LOW-LEVEL PROBLEM	HIGH-LEVEL PROBLEM
$\frac{2}{3} \times \frac{5}{8} = ?$	Create a real-world context for $\frac{2}{3} \times \frac{5}{8}$. Solve the problem without using the algorithm. Explain how your answer solves your problem.

Note how the low-level problem asks that students only apply a memorized algorithm. The high-level problem requires couching the problem in a real-world context, solving it by reasoning beyond an algorithm, and finally explaining how the computed solution really answers the real-world question.

Clearly it is up to the teacher to select or write problems and tasks that are high level and will engage their students. Effective tasks also provide multiple entry points for students. A task involving surveying students in the school to learn lunch preferences has many avenues for analyzing the data students collect. They could use graphical representations, compute various statistical values, or simply tally the opinions. Such a task has several entry points to begin and several effective solutions that students might produce.

3. USE AND CONNECT MATHEMATICAL REPRESENTATIONS An effective mathematics teaching practice focuses on many different representations of mathematical concepts. Children working to determine 8×7 might use a tally system with 7 groups of 8 bundles and count up; draw a rectangle that is 7 units by 8 units and count all the individual squares in the rectangle; use an array that is 7×8; or use a related number fact such as $8 \times 8 = 64$, and then reason that 7×8 is smaller than 64 by 8.

A common saying is that "If you only have a hammer, then every problem had better be a nail." In essence, if students have a single representation for mathematics to bring to a problem, it had better fit that particular problem, and the student had best be fluent with that single representation. An effective teaching practice helps students develop visual, physical, symbolic, contextual, and verbal representations to use as tools when they are engaged in mathematics tasks, allowing them to develop their personal way into a problem, rather than having to force the single representation they understand. According to NCTM (2014), "When students learn to represent, discuss, and make connections among mathematical ideas in multiple forms, they demonstrate deeper mathematical understanding and enhanced problem-solving abilities."

4. FACILITATE MEANINGFUL MATHEMATICAL DISCOURSE Authentic mathematics learning occurs when students have the opportunity to discuss their mathematics with the teacher and classmates. Students construct their personal understanding of mathematics by listening to, reasoning about, and discussing mathematics in the context of solving engaging tasks. Students can work independently on problems and questions, but real learning comes when they share their reasoning and solution strategies with others. Discourse involves exchanging ideas via a class discussion, employing visual, verbal, and/or written communication.

The norms surrounding such sharing should create an atmosphere where students can critique others' arguments and strategies without being critical of the students presenting them. Such an atmosphere allows for free discussions where methods and strategies are not tied directly to students' self-worth. As a result, students feel free to make mistakes, contribute to discussions, and listen to constructive comments of others.

5. POSE PURPOSEFUL QUESTIONS Some questions in mathematics class are meant to reveal what a student knows about a specific topic. Such questions do not produce meaningful discourse because the response is either right or wrong. A teacher can beneficially use such short-answer questions to determine student knowledge or to supply the answer to a class problem.

However, questions that require a simple numerical answer rarely lead to interesting mathematics discourse. Meaningful discourse is engendered by effective and purposeful questions. Questions that ask students to explain, describe, justify, or support require students to reason about the topic at hand and then to formulate a response that reveals their thinking and understanding. Such questions help students to make sense of what has been happening in class and can direct their thinking in ways that a simple question asking for the answer to a problem cannot. Questions that gather information are needed to establish what students know, while questions that encourage reflection and justification are essential to reveal student reasoning. Purposeful questions allow teachers to discern what students know and adapt lessons to meet varied levels of understanding, help students make important mathematical connections, and support students in posing their own questions (NCTM, 2014).

6. BUILD PROCEDURAL FLUENCY FROM CONCEPTUAL UNDERSTANDING Procedural fluency is different from procedural proficiency. Procedural proficiency suggests a student can use an algorithm such as long division quite well, but she may not understand how or why the division algorithm for whole numbers works. If she forgets one step in the algorithm, then she may not be able to solve a problem involving two-digit divisors. Additionally, a rote misapplication of an algorithm can often lead to curious results.

Procedural fluency suggests that the student is not only capable of using the algorithm, but she understands why the algorithm works and can explain it. Her understanding is based on conceptual tasks that preceded learning and using the algorithm. If she forgot a step in the algorithm, she could reconstruct the algorithm or solve the problem using the conceptual understanding tasks that preceded her learning the algorithm.

A student with procedural proficiency can select among various strategies to solve a problem, rather than apply by rote the same strategy or algorithm to all related problems. Such a student could solve $7 + 5 = ?$ by recalling number facts, or by regrouping to solve a related problem ($7 + 5 = 8 + 4$ or $6 + 6$).

7. SUPPORT PRODUCTIVE STRUGGLE IN LEARNING MATHEMATICS A mathematics class where students struggle seems like the antithesis of a successful class. The popular view of a model mathematics class suggests every student learning mathematics easily by listening to their teacher. Students who sit and listen do not learn meaningful mathematics. It is only by engaging with mathematics, by working through problems and applications that students really learn. The idea of a struggle with mathematics suggests students who wrestle with new concepts, work hard to connect what they are learning with previous knowledge, and persevere to make applications of their new mathematics.

Problems that promote struggling require time for students to engage mathematics problems and work out their responses. A question that elicits a response within five seconds is not a reasonable problem for helping students learn authentic mathematics. Framing better questions and tasks will help capture students' interest with their mathematics problems.

The key is to provide situations and contexts that are challenging but not frustrating. A frustrated student gives up and is less likely to invest time and effort in future challenging situations. In some cases teacher intervention by making a timely observation or suggestion can prevent frustration from building. Such interventions are not meant to be rescues where the teacher provides the entire path to a solution, but is an intervention that nudges students toward a productive path.

8. ELICIT AND USE EVIDENCE OF STUDENT THINKING The focus here is on student thinking as students construct their personal understanding of mathematics. Problems and tasks should be designed to reveal students' thinking and mastery of mathematics concepts. Certainly a quiz on Friday provides evidence of students' progress, but the information is too late to help correct students' erroneous information, poorly constructed relationships, or misconceptions. Lessons that precede Friday's quiz should include timely and frequent evidence gathering about what students know.

Student misconceptions can become entrenched and calcified if they are left untreated and unexposed for a period of time. A student who has been misapplying an algorithm for several days in mathematics class will have great difficulty correcting or deleting their misconception. If such a misconception were left to a quiz to reveal, it may prove very resilient and quite difficult to rectify or eradicate. Evidence of student thinking should be elicited daily such as by an exit ticket, and this information used to help students refine their thinking about a concept or application well in advance of a formal quiz or test.

> EXERCISE 1.2

In a classroom you are observing, give examples of how you see the eight principles at work or not in evidence.

<<<<<<<<<<<<<<<<<<<<<<<<<

1-2d Essential Elements

One final aspect of an effective mathematics program for schools concerns overarching elements that describe the entirety of the program from subject matter to the methods of teaching. In *Principles to Actions*, NCTM identifies how these five Essential Elements speak to factors in a successful mathematics curriculum that are not distinctly part of the curriculum or individual teaching design (see Table 1.3). The five Essential Elements are "essential elements of excellent mathematics program" (NCTM, 2014). The eight Mathematics Teaching Practices are impossible unless school mathematics programs have these five Essential Elements firmly in place (see Figure 1.4).

Many aspects of the Essential Elements involve district-wide features of an educational program. There are, however, aspects of each element that should be considered for individual classroom.

TABLE 1.3 Five Essential Elements of Effective School Mathematics Programs

Consistent implementation of effective teaching and learning of mathematics is possible only when school mathematics programs have in place:

1. A commitment to *access and equity*.
2. A powerful *curriculum*.
3. Appropriate *tools and technology*.
4. Meaningful and aligned *assessment.*
5. A culture of *professionalism*.

SOURCE: NCTM (2014).

Figure 1.4 The five Essential Elements